Cones

Mar 1

V a vector space. A <u>cone</u> $K \subseteq V$ is a set stable under positive multiplication (i.e. doesn't necessarily contain the origin).

The <u>dual cone</u> is defined

$$K^* = \{ \phi \in V^* \mid \phi(x) \geq 0 \ \ \forall x \in K \}$$

An extremal ray $r \subseteq K$ is an <u>extremal ray</u> if it is a one dimensional subcone s.t. if $x+y \in r$, then $x, y \in r$.

X a projective variety.
The <u>ample cone</u> $\mathrm{Amp}(X) \subset N^1(X)_{\mathbb{R}}$ is the cone of all ample $\mathbb{R}$-divisors.

The <u>nef cone</u> $\mathrm{Nef}(X) \subset N^1(X)_{\mathbb{R}}$ is the cone of all nef $\mathbb{R}$-divisors.

Notice: $\mathrm{Amp}(X)$ is (equivalently) the convex cone spanned by ample integral divisors.

By Kleiman's Thm...

<u>Thm</u>: $\mathrm{Nef}(X) = \overline{\mathrm{Amp}(X)}$ and $\mathrm{Amp}(X) = \mathrm{int}(\mathrm{Nef}(X))$.

<u>Pf</u>: We already know $\mathrm{Nef}(X)$ is closed and $\mathrm{Amp}(X)$ is open.

So $\overline{\mathrm{Amp}(X)} \subseteq \mathrm{Nef}(X)$ and $\mathrm{Amp}(X) \subseteq \mathrm{int}(\mathrm{Nef}(X))$

But if D is nef, H ample, we know $D + \varepsilon H$ is ample for suff small $\varepsilon > 0$, so $\mathrm{Nef} \subseteq \overline{\mathrm{Amp}}$.

Assume $D \subseteq \operatorname{int}(\mathrm{Nef})$. Then $D - \varepsilon H$ is nef for small $\varepsilon > 0$. Thus $(D - \varepsilon H) + \varepsilon H = D$ is ample. $\square$

<u>Cones of Curves</u>

Let $Z_1(X)_{\mathbb{R}}$ be the vector space of $\mathbb{R}$-linear combinations of irreducible curves on X.

$\gamma_1, \gamma_2 \in Z_1(X)_{\mathbb{R}}$ are <u>numerically equivalent</u> if $D \cdot \gamma_1 = D \cdot \gamma_2$ for every $D \in \mathrm{Div}_{\mathbb{R}}(X)$.

$$N_1(X)_{\mathbb{R}} := Z_1(X)_{\mathbb{R}} \big/ \text{numerical equivalence}$$

By construction, intersection $N^1(X)_{\mathbb{R}} \times N_1(X)_{\mathbb{R}} \to \mathbb{R}$ is a perfect pairing, so $N_1(X)_{\mathbb{R}} \cong N^1(X)_{\mathbb{R}}^*$.

The <u>cone of curves</u> $NE(X) \subseteq N_1(X)_{\mathbb{R}}$ is the cone spanned by non-negative linear combinations of curves, and

$\overline{NE}(X)$ is the <u>closed cone of curves</u>.

<u>Thm</u>: $\overline{NE}(X)$ is the cone dual to $\mathrm{Nef}(X)$. That is,

$$\overline{NE}(X) = \{\gamma \in N_1(X)_{\mathbb{R}} \mid D \cdot \gamma \geq 0 \ \forall \ D \in \mathrm{Nef}(X)\}$$

<u>Pf</u>: Exercise! (This is just checking cone stuff.)

We can test amplitude via cones:

<u>Thm:</u> $D \in \mathrm{Div}_{\mathbb{R}}(X)$ is ample

$\iff \overline{NE}(X) - \{0\} \subseteq D_{>0} := \{\gamma \in N_1(X) \mid D \cdot \gamma > 0\}$

<u>Pf:</u> Suppose the second statement holds.

Let $\varphi_D : N_1(X)_{\mathbb{R}} \to \mathbb{R}$ be intersection w/ D.

Then $\varphi_D(\gamma) > 0$ for all $\gamma \in \overline{NE} - \{0\}$.

Let $H_1, \dots, H_r$ be ample divisors that form a basis for $N^1(X)_{\mathbb{R}}$. Define the norm on $N_1(X)$:

$$\|\gamma\| = \sum |H_i \cdot \gamma|. \quad \text{(can check that this is actually a norm.)}$$

Let $S = \{\gamma \mid \|\gamma\| = 1\}$, the "unit sphere". Then $\varphi_D(\gamma) > 0$ for $\gamma \in \overline{NE}(X) - S$.

$\overline{NE}(X) - S$ is compact, so there is some $\varepsilon > 0$ s.t.

$\varphi_D(\gamma) \geq \varepsilon \quad \forall \gamma \in \overline{NE}(X) - S.$

Thus, $D \cdot C \geq \varepsilon \|C\| = \varepsilon(H \cdot C)$ for every irreducible curve $C \subseteq X$.

$\Rightarrow \dfrac{D \cdot C}{H \cdot C} \geq \varepsilon$, so D is ample.

Conversely, assume D is ample.

Let $\gamma \in \overline{NE}(X)$. Let $F \in N^1(X)$ s.t. $F \cdot \gamma < 0$.

Then $D + \varepsilon F$ is ample (and thus nef) for small ε, so $(D+\varepsilon F)\cdot\gamma \geq 0$
$\Rightarrow D \cdot \gamma \geq -\varepsilon F \cdot \gamma > 0 \Rightarrow \gamma \in D_{>0}$. $\square$

<u>Note</u>: Can use cones to see that if H is ample, there are only finitely many <u>integral</u> curve classes γ in $\overline{NE}(X)$ s.t. $H \cdot \gamma \leq M$ for some $M > 0$. <u>Exercise</u>: Prove this.

Structure of cones on a surface

X a sm. projective surface. Then

1.) $N_1(X)_{\mathbb{R}} = N^1(X)_{\mathbb{R}}$

2.) $\mathrm{Nef}(X) \subseteq \overline{NE}(X)$,
with equality $\Leftrightarrow C^2 \geq 0$ for all irreducible C.

<u>Pf</u>: D ample $\Rightarrow D = \sum a_i C_i$, C_i irr and $a_i > 0$, so $\mathrm{Amp}(X) \subseteq NE(X)$

Taking closures, get $\mathrm{Nef}(X) \subseteq \overline{NE}(X)$.

$\overline{NE}(X) \subseteq \mathrm{Nef}(X) \Leftrightarrow NE(X) \subseteq \mathrm{Nef}(X) \Leftrightarrow$ all irr C are nef
$\Leftrightarrow C^2 \geq 0 \ \forall$ irr C. $\square$

3.) Let $C \subset X$ irr. w/ $C^2 \leq 0$. Then $\overline{NE}(X)$ is spanned by C and $\overline{NE}(X)_{C\geq 0} := C_{\geq 0} \cap \overline{NE}(X)$.

Pf: If $C' \in NE(X)$ is effective and doesn't have C as a component, then $C' \cdot C \geq 0$, so $C' \in C_{\geq 0} \cap NE(X)$.

Thus, $NE(X) \subseteq \text{span}(C, C_{\geq 0} \cap \overline{NE}(X)) \subseteq \overline{NE}(X)$. $\square$

4.) If C is irreducible and $C^2 \leq 0$, then C is on the boundary of $\overline{NE}(X)$. If $C^2 < 0$, it's an extremal ray.

Pf: If $C^2 < 0$, then $C \notin NE(X)_{C\geq 0}$, so by 3.), C must be extremal.

If $C^2 = 0$, find $\gamma \in N_1(X)_{\mathbb{R}}$ s.t. $C \cdot \gamma < 0$.

Then $C(C + \varepsilon\gamma) = \varepsilon C \cdot \gamma < 0 \ \forall \ \varepsilon > 0$. Since C is nef, $C + \varepsilon\gamma \notin \overline{NE}(X)$, so C is on the boundary. $\square$

Mar 3

Claim: Let X be a sm. projective surface, and $\dim N^1(X)_{\mathbb{R}} > 1$. Suppose $C \subseteq X$ is an irreducible curve s.t. $C^2 > 0$. Then $C \in \text{int}(\overline{NE}(X))$.

Pf: By Hodge index, if $D \cdot C = 0$, then $D^2 < 0 \Rightarrow D$ is not nef.

So if D is nef $D \cdot C > 0$.

Choose a norm on $N^1(X)_{\mathbb{R}}$ and let $S \subseteq N^1(X)_{\mathbb{R}}$ be the corr. unit sphere.

For any $D \in \mathrm{Nef}(X) \cap S$, $\varphi_C(D) := D \cdot C > 0$. Thus, since $\mathrm{Nef}(X) \cap S$ is compact, there is some $M > 0$ s.t. $D \cdot C \geq M \quad \forall\, D \in \mathrm{Nef}(X) \cap S$.

Similarly, let $F \neq 0$ be some curve class in $N_1(X)_{\mathbb{R}}$. Then there is some $N > 0$ s.t. $|D \cdot F| \leq N \quad \forall\, D \in \mathrm{Nef}(X) \cap S$.

Take $\varepsilon \leq \frac{M}{2N}$, then for any $D \in \mathrm{Nef}(X) \cap S$,

$$D(C + \varepsilon F) = D \cdot C + \varepsilon D \cdot F \geq M - \frac{M}{2N} \cdot N = \frac{M}{2} > 0$$

So $C + \varepsilon F \in \overline{NE}(X)$.

Note: ε depends on F, but by convexity of $\overline{NE}(X)$, we only need to check C remains in $\overline{NE}(X)$ by moving it in $2 \cdot (\dim N_1(X)_{\mathbb{R}})$ many directions. $\square$

<u>Example: Cones of ruled surfaces</u>

C a smooth projective curve of genus g

E = rank 2 v.b. on C.

Let $X = \mathbb{P}(E)$, and $\pi : X \to C$ the projection.

To make computations easier, assume $\deg E$ is even.
X is invariant under tensoring by a l.b., so we can assume $\deg E = 0$.

$\mathcal{O}_X(1)$ = tautological quotient with respect to E.
(restricts as $\mathcal{O}_{\mathbb{P}^1}(1)$ on each fiber)

Have sequence $0 \to \mathcal{O}_X(-1) \longrightarrow \pi^* E \to \mathcal{O}_X(1) \longrightarrow 0$
(Euler sequence on each $\mathbb{P}^1$)

Recall:
$N^1(X)_{\mathbb{R}}$ is generated by $Z = c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$ and f = a fiber class

where $f^2 = 0$ and $Z \cdot f = \deg \mathcal{O}_f(1) = 1$.

Z^2 ?

Let $C_0 \subseteq X$ be a section.

$$\overset{\curvearrowleft}{\pi}$$

Then on C_0,

$$0 = \deg(\Lambda^2 E) = c_1(\mathcal{O}_X(1)) \cdot C_0 + c_1(\mathcal{O}_X(-1)) \cdot C_0$$

and on f, $\pi^* E$ is trivial, so

$$c_1(\mathcal{O}_X(-1)) \cdot f = -Z \cdot f$$

$$\Rightarrow c_1(\mathcal{O}_X(-1)) \equiv_{num} -Z$$

Thus, $z^2 = -c_1(\mathcal{O}_X(-1))z = c_2(\pi^* E) = \pi^*(c_2(E)) = 0$

no c_2 on a curve

So in general, $(af+bz)^2 = 2ab \Rightarrow \mathrm{Nef}(X) \subseteq$ 1st quadrant of f, z - plane

Since fibers are irreducible, and $f^2 = 0 \Rightarrow f$ is nef, so $\mathrm{Nef}(X)$ is in the 1st quadrant w/ f as one boundary.

To find other bonding ray, need to find $\overline{NE}(X)$.

<u>Case 1:</u> E unstable.

<u>Recall:</u> slope $E = \mu(E) = \frac{\deg(E)}{rk(E)}$

E is semi-stable $\Leftrightarrow \mu(L) \geq \mu(E)$ for any quotient $E \twoheadrightarrow L$

$\Leftrightarrow \mu(L) \leq \mu(E)$ for any sub $L \hookrightarrow E$

Otherwise, E unstable

<u>Exercise:</u> E a rank 2 bundle on a curve will be destabilized by a line bundle.

In our case, E unstable $\Leftrightarrow \exists$ a l.b. quotient $E \twoheadrightarrow A$ s.t. $\deg A < \frac{\deg E}{2} = 0$.

Define $C_0 = \mathbb{P}(A) \subset \mathbb{P}(E) = X$, where $\deg A = a < 0$

Then C_0 is a section, so $C_0 \cdot f = 1$, to find $C_0 \cdot Z$,

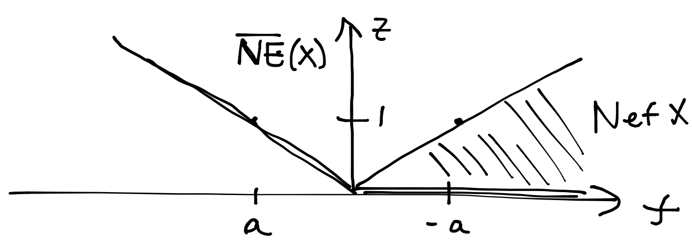
Consider $\pi^* E \to \mathcal{O}_X(1) \to 0$. Since $\mathcal{O}_X(1)$ is tautological, restriction to C_0 yields

$$E \longrightarrow A \to 0$$

so $C_0 \cdot Z = \deg A = a$.

Thus, $C_0 \equiv_{num} af + Z$, and $C_0^2 = 2a < 0$.

Thus, C_0 bounds $\overline{NE}(X)$, and it's dual to $(-af + Z)$, which is thus on the boundary of $\mathrm{Nef}(X)$, so we have



Mar 6

Case 2: E semi-stable.

Fact about stability:

a.) E, F semistable on a sm. curve $\Rightarrow E \otimes F$ semistable

b.) $E = E_1 \oplus E_2$ semistable $\Rightarrow E_i$ semistable.

Thus $E^{\otimes m}$ is semistable, and in characteristic 0, $S^m E$ is iso. to a summand of $E^{\otimes m}$, so $S^m E$ is semistable.

Can check: $\deg(S^m E) = 0$.

Suppose $\exists$ l.b. A on C of deg a s.t.

$H^0(S^m E \otimes A) \neq 0$

Then $0 \to \mathcal{O} \to S^m E \otimes A$

$\Rightarrow 0 \to A^* \to S^m E$.

Since $S^m E$ is semi-stable, $\deg A^* \leq 0 \Rightarrow a \geq 0$.

Suppose $C' \subset X$ is an effective curve.

Then C' is a section of $\mathcal{O}_X(m) \otimes \pi^* A$ for some $m \geq 0$ and A a l.b. on C.

If we pushforward $0 \to \mathcal{O}_X(-1) \to \pi^* E \to \mathcal{O}_X(1) \to 0$

$x \in C$

Base-change $\Rightarrow$ $R^i \pi_* (\mathcal{O}_X(-1))\big|_x \cong H^i(\underbrace{\pi^{-1}(x)}_{= \mathbb{P}^1}, \mathcal{O}(1)) \cong 0$

So
$$\begin{array}{l} \pi_* \pi^* E \cong \pi_* \mathcal{O}_X(1) \\ \quad \cong \\ \pi_* \mathcal{O}_X \otimes E \\ \quad \cong \\ \quad E \end{array}$$
(C is normal, and fibers of π are connected)

In fact, Exercise: $S^m E \cong \pi_* \mathcal{O}_X(m)$.

So $H^0(X, \mathcal{O}_X(m) \otimes \pi^* A) = H^0(C, \pi_* \mathcal{O}_X(m) \otimes A) = H^0(C, S^m E \otimes A)$

$\Rightarrow a = \deg(A) \geq 0$

So $C' = af + mZ$ lies in the first quadrant

$\Rightarrow \mathrm{Nef}(X) \subseteq NE(X) \subseteq$ quadrant 1

But their boundaries must be dual, so they both have boundary Z, and fill the 1st quadrant.

Exercise: Find $\overline{NE}(X)$ and $\overline{Nef}(X)$ for the following varieties:

1.) $X = \mathbb{P}^2$ blown up at 2 points

2.) $X = \mathbb{P}^3$ blown up at 3 collinear points

Question: In case 2 (E semi-stable), is $NE(X)$ closed?

i.e. is there some irr. curve Y s.t. $Y \equiv mZ$, some $m \geq 1$?

This is equivalent to C having a deg 0 line bundle A s.t.

$$H^0(S^m E \otimes A) \neq 0 \Rightarrow \underset{\mu=0}{A^*} \hookrightarrow \underset{\mu=0}{S^m E} \Rightarrow S^m E \text{ is semistable}$$

but not stable. But...

Thm (Hartshorne) If C has genus $g \geq 2$, then for any $r > 0$, $d \in \mathbb{Z}$, $\exists$ a strictly stable bundle E of rk r, deg d s.t. $S^m(E)$ is stable

$\forall$ m.

Thus, we can find an E of deg 0 s.t. there is no effective curve $Y \equiv mZ$ on $\mathbb{P}(E) \Rightarrow NE(X) \neq \overline{NE}(X)$

So in this situation, effective curves are (numerically) of the form $Y = aZ + bf$, where $a \geq 0$, $b > 0$.

Thus, $Z \cdot Y = b > 0$, so Z pairs positively with all effective curves but $Z^2 = 0$, so Z is not ample (which shows that amplitude can't be checked just by positivity on curves.)